Value at Risk: Assignment 1

Jake Mc Leroth (24913693)

10/03/2025

Contents

Introduction

The following report explores the use of parametric linear value at risk (VaR) models, based on chapter 2 of Alexander (2009). All methods are explained by example, with calculations done in Excel, Python and R. Throughout the report, any mention of a sheet, e.g. sheet Q2, is reference to a sheet in the Excel spreadsheet "assignment_1.xlsx"

Question 1: Normal Linear Value at Risk

In the normal linear VaR model, we make a few parametric assumptions to simplify the task of calculating VaR.

Firstly, we assume that portfolio returns are independent and identically distributed and follow a normal distribution (i.i.d normal). Since VaR is the value we are $100\alpha\%$ certain we won't exceed by a certain time in the future (Alexander, 2009), the i.i.d normal assumption implies that the expected return and volatility of returns are constant through time as well as returns will continue to follow an i.i.d normal distribution in the future. The convenience of this assumption is that it allows us to develop an analytic formula for VaR, making it easy to calculate. Furthermore, the i.i.d normal assumption allows us to scale VaR to any time period we want, using the square-root-of-time rule.

The second and third assumptions we make are that portfolio returns are a linear function of its assets or risk factors, and that the returns of these assets or risk factors have a multivariate normal distribution with a constant covariance matrix. Following this assumption, VaR can be decomposed into systematic Var and specific Var. Systematic VaR is the VaR captured by the risk factors, and Specific VaR is the remainder. Systematic VaR can then be disaggregated further into risk factor components, being done through either stand-alone VaR or marginal VaR. The decomposition that has followed from these two assumptions allows us to attribute levels of risk to the underlying risk factors, leading to more informed decisions. These decompositions will be explored more in the next section.

Question 2: Disaggregation of Systematic Value at Risk

In this question, we discuss systematic, stand-Alone and marginal VaR in the context of normal linear VaR. To do this, we look at example IV.2.5 from Alexander (2009). Consider portfolio with the following interest rate exposures and correlation matrix:

Interest Rate	US			UK		
Maturity (years)	1	2	3	1	2	3
PV01($)	1000	-1500	2000	800	900	-750
Volatility (bps)	100	80	70	85	75	65

Table 1: UK and US interest rate PV01 and volatilities

		US			UK		
		1	2	3	1	2	3
	1	1	0.95	0.90	0.70	0.67	0.62
US	2	0.95	1	0.97	0.65	0.75	0.75
	3	0.90	0.97	1	0.60	0.79	0.80
	1	0.70	0.65	0.60	1	0.98	0.95
UK	2	0.67	0.75	0.79	0.98	1	0.99
	3	0.62	0.75	0.80	0.95	0.99	1

Table 2: Interest Rate Correlations

It is important to note that since PV01 is in basis point terms, the volatilities are expressed in basis points.

Systematic VaR

According to Alexander (2009), the portion of returns that can be explained by the portfolio's risk factors is the systematic return. Hence, the systematic VaR is the VaR calculated from the systematic returns. It is calculated as follows:

$$\text{Systematic VaR}_{h,\alpha} = \Phi^{-1}(1-\alpha)\sqrt{\boldsymbol{\theta}'\boldsymbol{\Omega}_h\boldsymbol{\theta}} \tag{1}$$

Where

- $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_m)'$ is the vector of risk sensitivities
- $\boldsymbol{\Omega}_h$ is the h-day covariance matrix

Hence, $\sqrt{\boldsymbol{\theta}'\boldsymbol{\Omega}_h\boldsymbol{\theta}}$ is the h-day volatility of systematic returns. To continue with the example, we first need to calculate the covariance matrix. The diagonal elements are the risk factor volatilities squared, e.g. the value for US1 and US1 will be $100^2 = 10000$. The non-diagonal elements are the product of each risk factor's volatility as well as their shared correlation. For example, the value for US1 and US2 is $(100)(80)(0.95) = 7600$. The full calculation is shown in sheet Q2.

The annual covariance matrix is:

$$\mathbf{\Omega_{250}} = \begin{bmatrix} 10000 & 7600 & 6300 & 5950 & 5025 & 4030 \\ 7600 & 6400 & 5432 & 4420 & 4500 & 3900 \\ 6300 & 5432 & 4900 & 3570 & 4147.5 & 3640 \\ 5950 & 4420 & 3570 & 7225 & 6247.5 & 5248.75 \\ 5025 & 4500 & 4147.5 & 6247.5 & 5625 & 4826.25 \\ 4030 & 3900 & 3640 & 5248.75 & 4826.25 & 4225 \end{bmatrix} = \begin{bmatrix} \mathbf{\Omega}_{\text{US}} & \mathbf{\Omega}_{\text{US}-\text{UK}} \\ \mathbf{\Omega}'_{\text{US}-\text{UK}} & \mathbf{\Omega}_{\text{UK}} \end{bmatrix},$$

Where it is split up into its respective risk factor covariances ($\mathbf{\Omega}_{\text{US}}$ and $\mathbf{\Omega}_{\text{UK}}$ and cross-covariances $\mathbf{\Omega}_{\text{US}-\text{UK}}$). The vector of risk factor sensitivities is:

$$\boldsymbol{\theta}' = \left(\boldsymbol{\theta}'_{\text{US}}, \boldsymbol{\theta}'_{\text{UK}}\right),$$

Where

$$\boldsymbol{\theta}'_{\text{US}} = (1000, -1500, 2000), \quad \boldsymbol{\theta}'_{\text{UK}} = (800, 900, -750)$$

The variance of systematic returns is thus:

$$\boldsymbol{\theta}'\mathbf{\Omega}_h\boldsymbol{\theta} = 35,519,275,000$$

And, hence, using the square-root-of-time rule, we get the 1% 10-day systematic normal linear VaR:

$$\Phi^{-1}(0.99)(\sqrt{35,519,275,000})(\sqrt{\frac{10}{250}}) = \$87,687.30.$$

Stand-Alone VaR

The stand-alone VaR is the systematic VaR from different risk factors (Alexander, 2009). For instance, we can work out equity VaR, which is the systematic VaR due to exposure to equity. The method of calculating stand-alone VaR is to calculate systematic VaR as shown previously; however, we set all the values in the risk sensitivity vector that we aren't interested in to zero. In our example, this means setting the UK values to zero. This is equivalent to just using $\boldsymbol{\theta}_{\text{US}}$ and $\mathbf{\Omega_{US}}$ for the calculation. This is shown as follows:

$$\begin{aligned} \text{1\% 10-day US Interest Rate VaR} &= \Phi^{-1}(0.99)(\sqrt{\boldsymbol{\theta}'_{\text{US}}\mathbf{\Omega_{US}}\boldsymbol{\theta}_{\text{US}}})(\sqrt{\frac{10}{250}}) \\ &= \$54,672.64 \end{aligned}$$

Similarly for UK interest rate VaR:

$$\begin{aligned} \text{1\% 10-day UK Interest Rate VaR} &= \Phi^{-1}(0.99)(\sqrt{\boldsymbol{\theta}'_{\text{UK}}\mathbf{\Omega_{UK}}\boldsymbol{\theta}_{\text{UK}}})(\sqrt{\frac{10}{250}}) \\ &= \$40,931.23 \end{aligned}$$

We can see that the sum of the stand-alone VaRs is larger than the systematic VaR. This is as a result of the correlation between the US and UK interest rates that has not been accounted for (i.e. the diversification effect). The systematic VaR will only equal the sum of the stand-alone VaRs if all the risk factors have perfect correlation.

Marginal VaR

Marginal VaR is also a disaggregation of systematic VaR, however, unlike stand-alone VaR, it adds up to the systematic VaR (Alexander (2009)). It is calculated as follows:

$$\text{Marginal VaR} \approx \boldsymbol{\theta}'\mathbf{g}(\boldsymbol{\theta}) \tag{2}$$

Where

$$\mathbf{g}(\boldsymbol{\theta}) = \Phi^{-1}(1-\alpha)\left(\boldsymbol{\Omega}_h\boldsymbol{\theta}\right)\left(\boldsymbol{\theta}'\boldsymbol{\Omega}_h\boldsymbol{\theta}\right)^{-1/2}.$$

$\mathbf{g}(\boldsymbol{\theta})$ is the derivative of VaR with respect to $\boldsymbol{\theta}$. To calculate $\mathbf{g}(\boldsymbol{\theta})$ we need $\boldsymbol{\theta}'\boldsymbol{\Omega}_h\boldsymbol{\theta}$, which we calculated previously, and $\boldsymbol{\Omega}_h\boldsymbol{\theta}$. In Sheet Q2 we get the following:

	$\boldsymbol{\Omega}_h\boldsymbol{\theta}$	$\mathbf{g}(\boldsymbol{\theta})$
US1	17,460,000	215.5196391
US2	13,525,000	166.9474868
US3	11,810,750	145.7874329
UK1	13,926,187.5	171.8995936
UK2	13,010,812.5	160.6005507
UK3	10,833,875	133.7292572

Table 3: Table of $\boldsymbol{\Omega}_h\boldsymbol{\theta}$ and $\mathbf{g}(\boldsymbol{\theta})$ values

The marginal Var is then calculated as follows:

$$1\% \text{ annual US marginal VaR} = \boldsymbol{\theta}'_{\text{US}}\mathbf{g}\left(\boldsymbol{\theta}_{\text{US}}\right) = \$256,673.27$$

$$\Rightarrow 1\% \text{ 10-day US marginal VaR} = (\$256,673.27)\left(\sqrt{\frac{10}{250}}\right) = \$51,334.65.$$

The 1% 10-day UK marginal VaR is then calculated as \$ 36,352.65. We see that this sums to the systematic VaR, with the US accounting for around 59% of the systematic VaR. The process does not need to stop here—you can further calculate the marginal VaRs of the individual interest rate exposures.

Question 3: Principal Components Value at Risk

According to Alexander (2008), principal component analysis (PCA) is an orthogonalization technique that turns a set of correlated variables into a smaller set of uncorrelated variables. Essentially, the goal of PCA is to reduce dimensionality while keeping as much of the information as possible. To start, we decompose the covariance matrix:

$$\mathbf{\Omega} = \mathbf{W}\mathbf{\Lambda}\mathbf{W}',$$

Where

- $\mathbf{W}$ is a matrix of n eigenvectors (principal components) ($\mathbf{\Omega}$ is $n \times n$)
- $\mathbf{\Lambda}$ is a diagonal vector of eigenvalues, showing the variance of each principle component (in descending order)

In sheet Q3 Returns, we have the returns from 60 different bond maturities used in examples IV.2.9 and IV.2.10 in Alexander (2009). Using the matrix.xla add-in, we get the 60x60 covariance matrix in sheet Q3 Covariance, which we then use to get the PCA decomposition in sheet Q3 PCA. We see from this sheet that the first principal component explains around 93% of the variance.

To use PCA for VaR (or anything for that matter), we can select the first k principal components to simplify the calculation. In the case study in Alexander (2009), the first three principal components are used. As we saw in sheet Q3 PCA, the first three principal components explain more than 99% of the variation. To calculate the VaR, we get the first three principal component sensitivities by multiplying the PV01 vector with a matrix consisting of the first three columns of $\mathbf{W}$ (i.e. the first three principal components). That gives us the following:

Component	**PC1**	**PC2**	**PC3**
Net Sensitivities	428.1480022	-2974.58559	1041.210828

Table 4: Net Sensitivities to Principal Components

This is now our PCA version of the risk factor sensitivity vector. We then use a matrix consisting of the first three components of $\mathbf{\Lambda}$, which tell us the variances of the first three principal components:

$$\begin{bmatrix} 856.8228714 & 0 & 0 \\ 0 & 45.29997223 & 0 \\ 0 & 0 & 9.152670483 \end{bmatrix}.$$

Doing matrix multiplication, we get the PCA equivalent of $\boldsymbol{\theta}'\mathbf{\Omega}_h\boldsymbol{\theta}$. The PCA VaR is then calculated in sheet Q3 PCA VaR, resulting in the value of £175,297.50. This is very close to the true value of £176,549. The advantage of this method is that the dimension reduction is much more computationally efficient—the matrix multiplication required for calculating VaR can become extremely time-consuming from a computational perspective when our covariance matrices become very large.

Question 4: Foreign Currency Systematic Value at Risk

In this question, we look at examples IV.2.14 and IV.2.15 from Alexander (2009) to extend our knowledge of disaggregating systematic VaR to an equity portfolio with offshore investments (i.e. forex exposure). The idea is that portfolio returns can be decomposed into the returns on the indices it's invested in and the returns on the foreign currency it's exposed to. The risk factor sensitivity vector consists of a sub-vector of market betas and a sub-vector of ones for the forex exposures.

First consider exposure to one foreign currency: An investor from the US invests \$2 million in a selection of FTSE 100 equities with the following information:

- Portfolio beta is 1.5
- FTSE Volatility is 15%
- The \$/£ exchange rate volatility is 20%
- The correlation is 0.3 (this is referred to as "quanto correlation")

The risk factor sensitivity vector is then $\boldsymbol{\theta}' = (1.5, 1)$. We want to calculate the 1% 10-day systematic VaR. Using the same methodology as before, we get the annual covariance matrix:

$$\boldsymbol{\Omega_{250}} = \begin{bmatrix} 0.0225 & 0.009 \\ 0.009 & 0.04 \end{bmatrix}$$

And thus by multiplying each value by $\frac{10}{250}$ we get the 10-day covariance matrix:

$$\boldsymbol{\Omega_{10}} = \begin{bmatrix} 0.0009 & 0.00036 \\ 0.00036 & 0.0016 \end{bmatrix}.$$

The systematic VaR is now easily calculated:

$$\begin{aligned} \text{Systematic VaR}_{10,0.01} &= \Phi^{-1}(0.99)\sqrt{\boldsymbol{\theta}'\boldsymbol{\Omega}_{10}\boldsymbol{\theta}} \\ &= 15.9571186\%. \end{aligned}$$

This translates to a dollar value of \$319,142.37. The stand-alone VaRs are easily calculated with equity VaR $= (1.5)(2,000,000)(0.15)(\sqrt{\frac{10}{250}})(\Phi^{-1}(0.99)) =$ \$209,371.31 and forex VaR $= (1)(2,000,000)(0.2)(\sqrt{\frac{10}{250}})(\Phi^{-1}(0.99)) =$ \$186,107.83. In sheet Q4 a), we get the following:

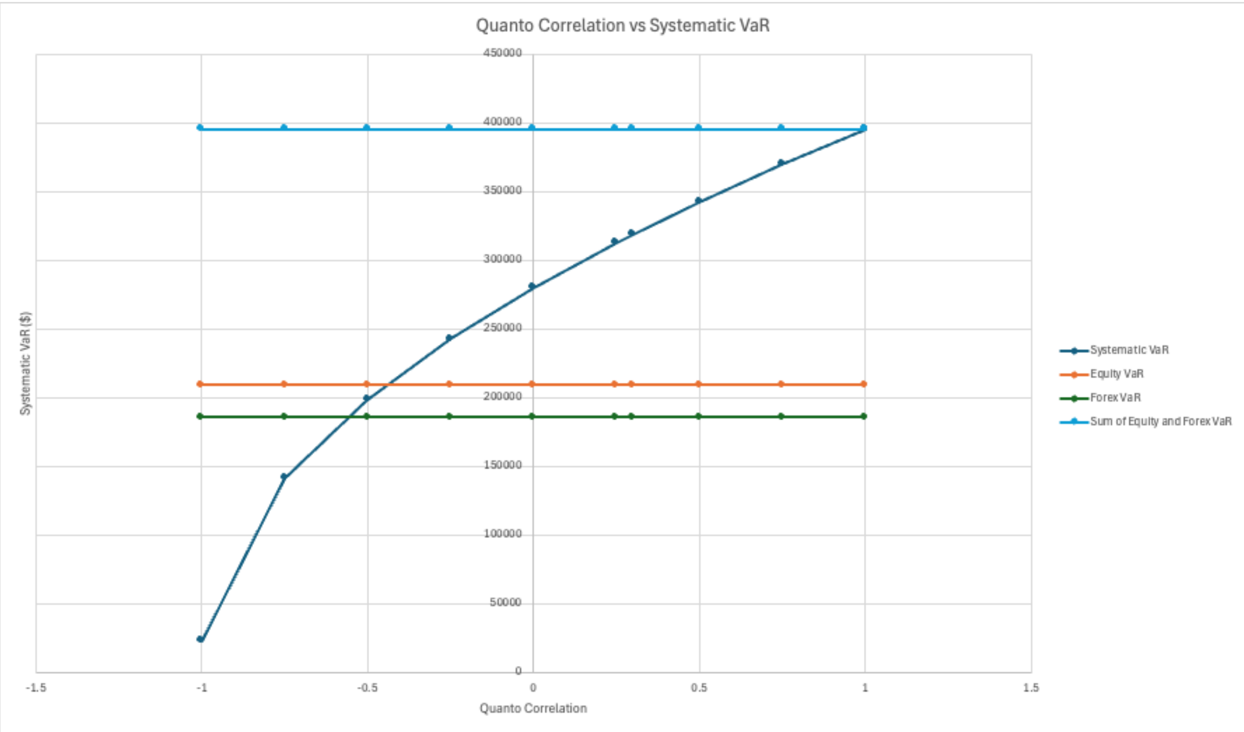
	$\boldsymbol{\Omega_{10}\theta}$	$\boldsymbol{g(\theta)}$
FTSE 100	0.00171	0.057995054
\$/£	0.00214	0.072578605

Table 5: Table of $\boldsymbol{\Omega_{10}\theta}$ and $\boldsymbol{g(\theta)}$ values

The marginal VaRs are therefore:

$$\begin{aligned}
\text{Eqity marginal VaR} &= (1.5)(2,000,000)(0.057995054) \\
&= \$173,985.16 \\
\text{Forex marginal VaR} &= (1)(2,000,000)(0.072578605) \\
&= \$145,157.21 \\
\text{Sum of Marginals} &= \$319,142.37
\end{aligned}$$

The effect of the quanto correlation is shown in the following graph:



The graph shows that when there is a strong negative quanto correlation, the total systematic VaR can actually be less than the stand-alone VaRs (Alexander, 2009).

Example IV.2.15 now extends this methodology to multiple equity index and forex exposures. Consider the following:

Index	**Amount (\$)**	**Beta**	**Return**	**Vol**	**Net Dollar Beta**
S&P 500	2,000,000	0.9	X_1	0.2	1,800,000
FTSE 100	2,000,000	1.1	X_2	0.22	2,200,000
CAC 40	3,000,000	1.2	X_3	0.25	3,600,000
DAX 30	4,000,000	1.3	X_4	0.27	5,200,000

Table 6: Local Exposures Table

Exposure	**Return**	**Vol**	**Amount (\$)**
\$/£	X_5	0.15	2,000,000
\$/	X_6	0.10	7,000,000

Table 7: Forex Exposures Table

Since there are different dollar values exposed to each index, it is easier to work with $\boldsymbol{\theta}$ in dollar terms:

$$\boldsymbol{\theta} = \begin{bmatrix} 1,800,000 \\ 2,200,000 \\ 3,600,000 \\ 5,200,000 \\ 2,000,000 \\ 7,000,000 \end{bmatrix}.$$

This means the portfolio's return is equal to $1000000(1.8X_1 + 2.2X_2 + 3.6X_3 + 5.2X_4 + 2X_5 + 7X_6)$. The following correlation matrix is also given:

	S&P 500	**FTSE 100**	**CAC 40**	**DAX 30**	**\$/£**	**\$/€**
S&P 500	1.00	0.75	0.75	0.75	0.20	0.20
FTSE 100	0.75	1.00	0.75	0.75	0.20	0.20
CAC 40	0.75	0.75	1.00	0.75	0.20	0.20
DAX 30	0.75	0.75	0.75	1.00	0.20	0.20
\$/£	0.20	0.20	0.20	0.20	1.00	0.50
\$/€	0.20	0.20	0.20	0.20	0.50	1.00

Table 8: Correlation Matrix

In the sheet Q4 b), we convert this into an annual covariance matrix:

$$\boldsymbol{\Omega}_{250} = \begin{bmatrix} 0.0400 & 0.0330 & 0.0375 & 0.0405 & 0.0060 & 0.0040 \\ 0.0330 & 0.0484 & 0.04125 & 0.04455 & 0.0066 & 0.0044 \\ 0.0375 & 0.04125 & 0.0625 & 0.050625 & 0.0075 & 0.0050 \\ 0.0405 & 0.04455 & 0.050625 & 0.0729 & 0.0081 & 0.0054 \\ 0.0060 & 0.0066 & 0.0075 & 0.0081 & 0.0225 & 0.0075 \\ 0.0040 & 0.0044 & 0.0050 & 0.0054 & 0.0075 & 0.0100 \end{bmatrix}$$

Since we have $\boldsymbol{\theta}$ and $\boldsymbol{\Omega}_{250}$, we make use of the same methodology as before, resulting in the following:

$$\begin{aligned}
\text{1\% 10-day sytematic VaR} &= \$1,490,889.25 \\
\text{Equity stand-alone VaR} &= \$1,333,847.29 \\
\text{Forex stand-alone VaR} &= \$413,540.64 \\
\text{Equity marginal VaR} &= \$6,423,823.06 \\
\text{Forex marginal VaR} &= \$1,030,623.21.
\end{aligned}$$

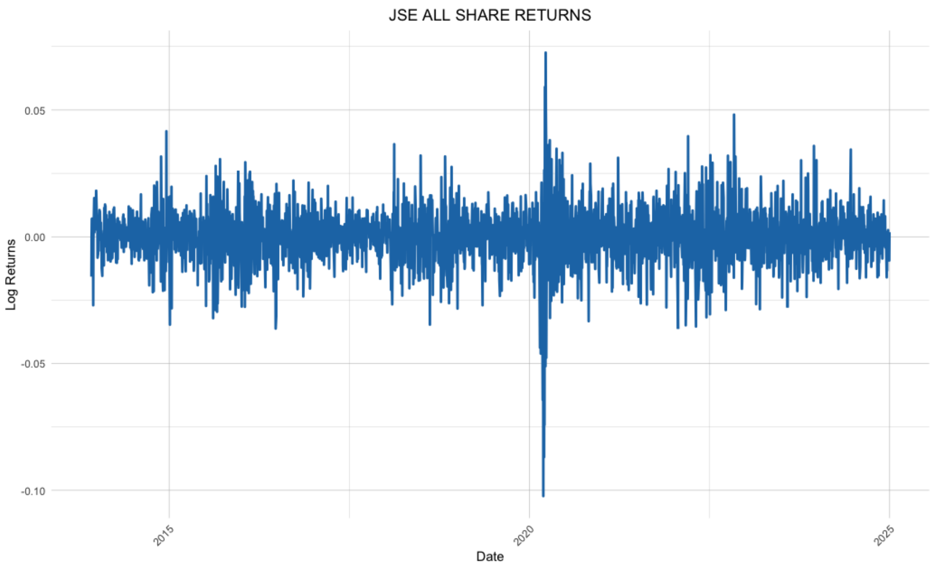
We see that marginal equity VaR accounts for approximately 86% of the systematic VaR.

Question 5: Parametric Alternatives to Normal Linear Value at Risk

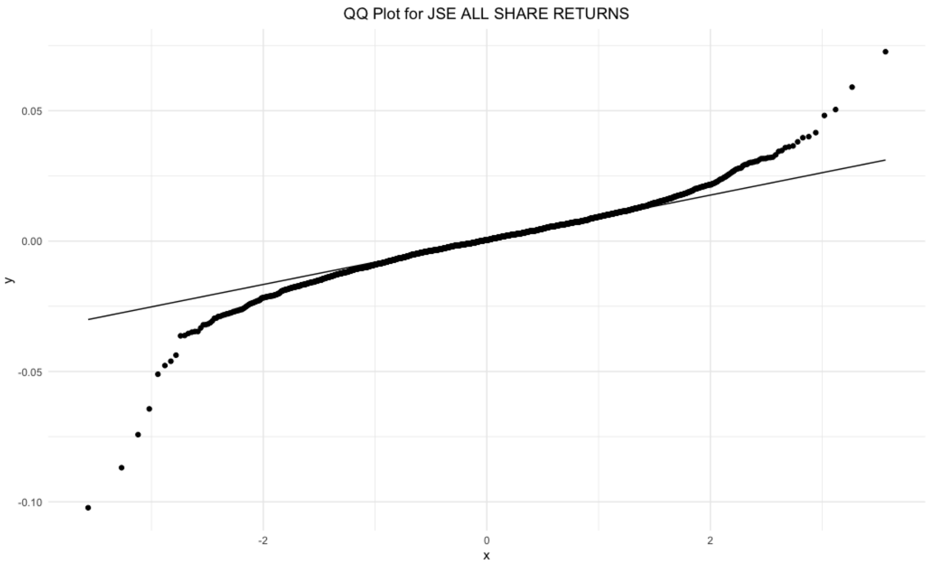
This question explores two parametric alternatives to the normal linear VaR model, namely normal mixture Linear VaR and student t-distributed linear VaR. According to Alexander (2009), return distributions tend to be leptokurtic and negatively skewed. This means that using the normal distribution may be underestimating VaR. The mixture distribution allows us to capture this skewness by modelling returns as different regimes, for example, a high volatility regime and a low volatility regime. The student t-distribution allows us to model the leptokurtic nature of returns. To start, we give some empirical evidence from South African indices that returns are not necessarily normally distributed. We look at the JSE All Share Index (ALSI), the JSE Top 40 Index (Top40) and the JSE Industrial 25 Index (Indi25).

a) Normality

We start by calculating the log returns and using a Q-Q plot to compare the sample quantiles to the theoretical quantiles of the normal distribution. This is done in the R file "q5_ab.R". The ALSI returns are seen over time:



The Q-Q plot for the ALSI is shown here:



We can very easily see that the tails of the return distribution do not match those of the theoretical normal distribution. The returns and Q-Q plots for Top40 and Indi25, which are shown in figures 1 to 4 in the appendix, show the same tail behaviour. The return distributions themselves also show that some periods seem to have higher volatility than others. To ensure that the distributions are in fact non-normal, we make use of the following statistical tests, all implemented in the R file:

Table 9: Normality Test Results for JSE Returns

Test	**JSE ALL SHARE**	**JSE TOP 40**	**JSE INDUSTRIAL 25**
Kolmogorov–Smirnov	$p = 3.051 \times 10^{-8}$	$p = 2.479 \times 10^{-7}$	$p = 3.402 \times 10^{-5}$
Shapiro-Wilk	$p < 2.2 \times 10^{-16}$	$p < 2.2 \times 10^{-16}$	$p < 2.2 \times 10^{-16}$
Anderson-Darling	$p < 2.2 \times 10^{-16}$	$p < 2.2 \times 10^{-16}$	$p < 2.2 \times 10^{-16}$

The null hypothesis for the KS test and the AD test (Alexander, 2008), as well as the SW test (Cryer and Chan, 2008), is that the distribution is normal. Hence, we see for all tests on all indices the returns are not normal at a 5% significance level (or ven a 1% significance). This shows the normal assumption is not valid for the South African indices.

b) Independence

Next, we check if the returns are independent. To do this, we check for autocorrelation up to lag 4 using the Ljung–Box Test as given in Cryer and Chan (2008). The results from R are given in the following table:

Test	**X-squared**	**Degrees of Freedom (df)**	**p-value**
All Share Returns	6.8402	4	0.1446
Top 40 Returns	6.9086	4	0.1408
Industrial 25 Returns	2.012	4	0.7336

Table 10: Box–Ljung Test Results

In the Ljung–Box test, the null hypothesis is that the observations are uncorrelated (Cryer and Chan, 2008). We see from the p-values that we fail to reject the null hypothesis at a 5% significance level. Hence, all the return series' seem to be independent. If they were not independent, we could use the following adjustment given in Alexander (2009):

$$\mathrm{VaR}_{\mathrm{h},\alpha} = \sqrt{\tilde{h}}\Phi^{-1}(1-\alpha)\sigma_1$$
$$\tilde{h} = h + 2\frac{\varrho}{(1-\varrho)^2}\left[(h-1)(1-\varrho) - \varrho\left(1-\varrho^{h-1}\right)\right]$$

Where ϱ is the autocorrelation between successive returns. Based on subsections a) and b), using a student t-distribution would be better to better capture the behaviour of the tails. Looking at the returns over time, however, we see that volatility is higher during some periods. This shows that a mixture distribution may be better account for different market regimes.

Normal Mixture and t-Distribution Value at Risk

We estimate the parameters of the normal mixture distributions for the ALSI, Top40 and Indi25 returns using the GaussianMixture object from scikit-learn in the "q5_em_jse.py" python script. In the script, we fit a normal mixture to each return series 100 times, and then take the best fit by looking at the aic, bic and log-likelihood - the values from these tests can be found in the appendix. The GaussianMixture object makes us of the EM algorithm that is explained in Alexander (2008). We then fit a t-distribution to all three return series using the maximum likelihood method in the spreasheet "q5_t_dist.xlsx". The estimated values are as follows:

Table 11: Normal Mixture Parameter Estimates

	π	μ_1	μ_2	σ_1	σ_2
ALSI	35.52%	-0.0775%	0.0782%	1.5746%	0.6983%
TOP40	35.66%	-0.0721%	0.0757%	1.6600%	0.7530%
INDI25	38.92%	-0.1035%	0.1148%	1.6212%	0.8010%

Table 12: t-Distribution Parameter Estimates

	Degrees of Freedom	μ	σ
ALSI	4.29	0.023%	1.091%
TOP40	4.58	0.023%	1.159%
INDI25	3.99	0.030%	1.190%

Normal mixture VaR is calculated using the following equation:

$$\pi P\left(Y_1 < (x_\alpha - \mu_1)\,\sigma_1^{-1}\right) + (1-\pi)P\left(Y_2 < (x_\alpha - \mu_2)\,\sigma_2^{-1}\right) = \alpha$$

Where $\mathrm{VaR}_{h,\alpha} = -x_{h,\alpha}$. This is calculated using solver in sheet Q5. Student t VaR is also calculated in the spreadsheet, using the equation:

$$\text{Student t}\,\mathrm{VaR}_{h,\alpha,v} = \sqrt{v^{-1}(\nu-2)h}t_v^{-1}(1-\alpha)\sigma - h\mu$$

The results for the 1-day normal mixture and student t VaRs:

Table 13: Value at Risk (VaR) Comparison

	Normal mixture VaR	Student tVaR
ALSI	3.0831%	2.97%
TOP40	3.2436%	3.24%
INDI25	3.2626%	3.79%

Note that in this example we have not taken expected retrun to be equal to the discount rate. Hence we subtract the expected return.

Conclusion

This report has illustrated how to calculate VaR, as well as its decompositions, in various different scenarios. It has specifically highlighted the importance of checking model assumptions and evaluating model risk.

References

Alexander, C. (2008). *Market risk analysis, quantitative methods in finance.* John Wiley & Sons.

Alexander, C. (2009). *Market risk analysis, value at risk models.* John Wiley & Sons.

Cryer, J.D. and Chan, K.-S. (2008). *Time Series Analysis with Applications in R.* 2nd edn. Springer, New York. ISBN 978-0-387-77316-2.

Appendix

Figure 1

Table 14: Best Model for ALSI

Metric	Value
Means	[0.00078161, -0.00077459]
Variances	[4.87674198e-05, 2.47948999e-04]
Weights	[0.64477517, 0.35522483]
BIC	-17426.84
AIC	-17456.46
Log-likelihood	3.16

Figure 2

Table 15: Best Model for TOP40

Metric	Value
Means	[-0.0007205, 0.00075729]
Variances	[2.75564762e-04, 5.66977803e-05]
Weights	[0.3566325, 0.6433675]
BIC	-17062.31
AIC	-17091.93
Log-likelihood	3.09

Table 16: Best Model for INDI25

Metric	Value
Means	[-0.00103481, 0.00114812]
Variances	[2.62833222e-04, 6.41540438e-05]
Weights	[0.38922774, 0.61077226]
BIC	-16830.29
AIC	-16859.92
Log-likelihood	3.05

Figure 3

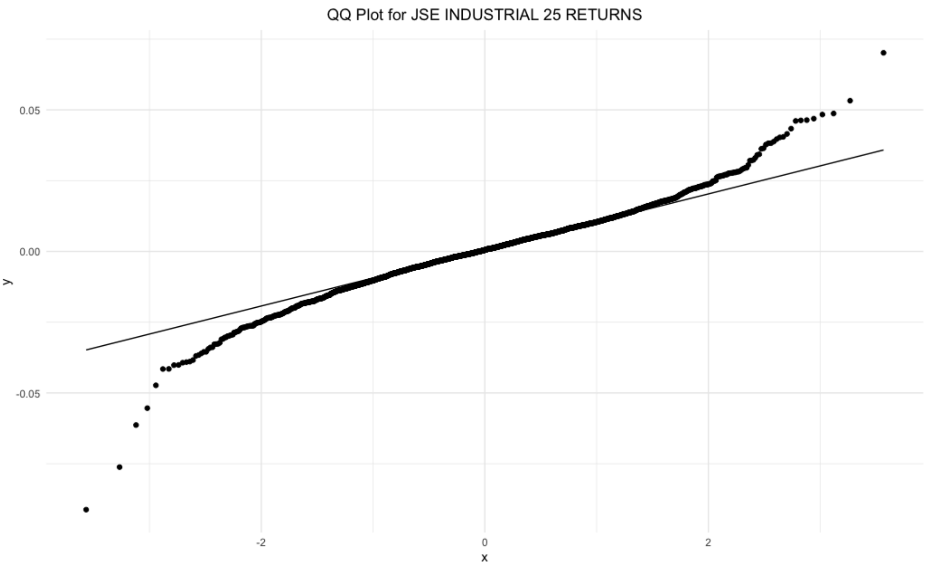


Figure 4